

Data Envelope analysis

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Lecture 3

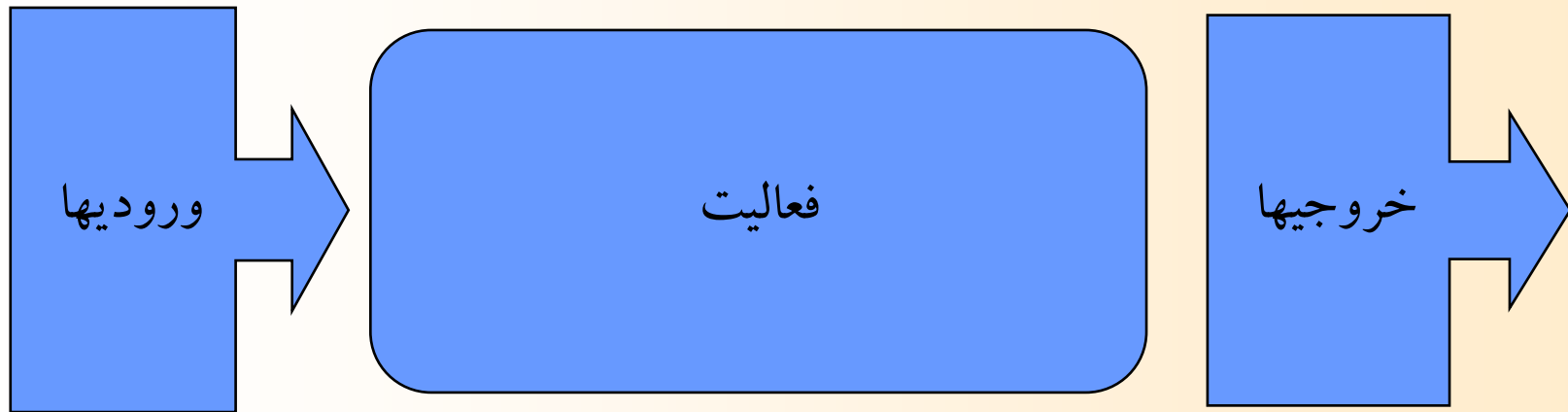
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بررسی کارایی

- Efficiency analysis has always interested researchers because of the relative difficulty encountered in assessing the performance of a firm or an organization. Using an engineering-like approach, Farrell (1957) attempted to measure the efficiency of a unit of production in the single input-single output case. Farrell's study involved the measurement of price and technical efficiencies and the derivation of the efficient production function
- Farrell applied his model to estimate the efficiency of the US agriculture relative to other countries. However, he failed in providing a way to summarize all various inputs and outputs into a single virtual input and single virtual out

بررسی ورودی و خروجی



چگونه کارایی فعالیت را در مورد تبدیل
چندین ورودی به چندین خروجی را میتوان اندازه گیری نمود

کارایی فنی یا بهره وری

- Charnes, Cooper and Rhodes extended Farrell's idea and proposed a model that generalizes the single-input, single-output ratio measure of efficiency of a single Decision-Making Unit (DMU) in a multiple-inputs, multiple outputs setting.
- **A DMU is an entity that produces outputs and uses up inputs.** In banking, a bank branch constitutes a DMU. The technical efficiency of a DMU is computed using the engineering-like efficiency measure of efficiency as ratio of virtual output produced to virtual input consumed:

$$\text{Technical efficiency} = \frac{\sum \text{weighted outputs}}{\sum \text{weighted inputs}}$$

DEA & Regression مقایسه

- ***As for the weights used in the transformation of the vectors of inputs and outputs into two single virtual scalars, the DEA model allows each DMU to choose the set of multipliers (weights) μ_o and ν_o that permits it to appear in the best light.***
- ***The efficiency score obtained is also relative to a sample of DMUs under analysis since the set of weights has to be feasible for other units and none of these units should have an efficiency score greater than one.***
- ***In contrast to Regression Analysis, which gives us an average profile of DMUs under analysis , DEA yields a piecewise empirical external production surface that, in economic terms represents the revealed best practice production frontier (or envelope).***
- ***By projecting each unit onto the frontier, it is possible to determine the level of inefficiency by comparison to a single reference unit or a convex combination of other reference units.***
- ***The projection refers to a virtual DMU which is a convex combination of one or more efficient DMUs. Thus, the projected point may itself not be an actual DMU***

معرفي DEA

- Data envelopment analysis (DEA), occasionally called frontier analysis, was first put forward by Charnes, Cooper and Rhodes in 1978. It is a performance measurement technique which can be used for evaluating the relative efficiency of decision-making units (DMU's) in organisations.
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- Examples of such units to which DEA has been applied are: banks, police stations, hospitals, tax offices, prisons, defence bases (army, navy, air force), schools and university departments.

تعيين ورودي و خروجي

- There is an increasing concern with measuring and comparing the efficiency of organisational units such as local authority departments, schools, hospitals, shops, bank branches and similar instances where there is a relatively homogeneous set of units.
 - The usual measure of efficiency, i.e.:
- $$\text{efficiency} = \frac{\text{output}}{\text{input}}$$
- is often inadequate due to the existence of multiple inputs and outputs related to different resources, activities and environmental factors. This problem can be illustrated for depots of a large retailing organisation which distributes goods to supermarkets. In this case the inputs for an efficiency measure are taken to be the value of the stock and the recurrent costs mainly in the form of wages, as those are resources supporting the operation of the depots.
 - Input: Stock is reasonably an input to an efficiency measure as an efficient depot will attempt to give a good service at lower stock levels, saving on capital and space.
 - Output: The outputs correspond to activities of the depots and are measured by the number of issues representing deliveries to supermarkets, the number of receipts in bulk from suppliers, and the number of requisitions on suppliers where they are out of stock or approaching stock out.
 - (A further possible output would be a measure of the service level).

بهره وري چند عاملي

- The measurement of relative efficiency where there are multiple possibly incommensurate inputs and outputs was addressed by Farrell and developed by Farrell and Fieldhouse(2), focusing on the construction of a hypothetical efficient unit, as a weighted average of efficient units, to act as a comparator for an inefficient unit.
- A common measure for relative efficiency is,

$$\text{Efficiency} = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}$$

$$\text{Efficiency of unit } j = \frac{u_1 y_{1j} + u_2 y_{2j} + \dots}{v_1 x_{1j} + v_2 x_{2j} + \dots}$$

where

- u_1 = the weight given to output 1
- y_{1j} = amount of output 1 from unit j
- v_1 = weight given to input 1
- x_{1j} = amount of input 1 to unit j.

مشكلات يافتن شاخص بهره وري

- The initial assumption is that this measure of efficiency requires a common set of weights to be applied across all units. This immediately raises the problem of how such an agreed common set of weights can be obtained. There can be two kinds of difficulties in obtaining a common set of weights
- First of all it may simply be difficult to value the inputs or outputs
- Alternatively different depots may choose to organise their operations differently so that the relative values of the different outputs may legitimately be different

مدل ریاضی

$$\begin{aligned} \text{Max } h_0 &= \frac{\sum_r u_r y_{rj_0}}{\sum_i v_i X_{ij_0}} \\ \text{subject to} & \\ & \frac{\sum_r u_r y_{rj}}{\sum_i v_i X_{ij}} \leq 1 \quad \text{for each unit } j. \\ & u_r, v_i \geq \varepsilon \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{Max } h_0 &= \frac{\sum_r u_r y_{rj_0}}{\sum_i v_i X_{ij_0}} \\ \text{subject to} & \\ & \frac{\sum_r u_r y_{rj}}{\sum_i v_i X_{ij}} \leq 1 \quad \text{for each unit } j. \\ & u_r, v_i \geq \varepsilon \end{aligned}} \right\} \text{(M1)}$$

مثال

	موجودي	دستمزد	اقلام	عمليات دريافت	عمليات خروج
انتبار ۱	3	5	40	55	30
انتبار 2	2.5	4.5	45	50	40
انتبار 3	4	6	55	45	30
انتبار 4	6	7	48	20	60
انتبار S	2.3	3.5	28	50	25
انتبار 6	4	6.5	48	20	65
انتبار 7	7	10	80	65	57
انتبار 8	4.4	6.4	25	48	30
انتبار 9	3	5	45	64	42
انتبار 10	5	7	70	65	48
انتبار 11	5	7	45	65	40
انتبار 12	2	4	45	40	44
انتبار 13	5	7	65	25	35
انتبار 14	4	4	38	18	64
انتبار 15	2	3	20	50	15
انتبار 16	3	6	38	20	60
انتبار 17	7	11	68	64	54
انتبار 18	4	6	25	38	20
انتبار 19	3	4	45	67	32
انتبار 20	5	6	57	60	40

مثال

$$\begin{aligned}
 \text{Max } h_0 &= \frac{40u_1 + 55u_2 + 30u_3}{3v_1 + 5v_2} \\
 \text{subject to} \\
 \varepsilon \frac{40u_1 + 55u_2 + 30u_3}{3v_1 + 5v_2} &\leq 1 \quad (\text{depot 1}) \\
 \frac{45u_1 + 50u_2 + 40u_3}{2.5v_1 + 4.5v_2} &\leq 1 \quad (\text{depot 2}) \\
 \dots\dots\dots &\text{for remaining depots} \\
 \text{and } u_1, u_2, v_1, v_2 &\geq \varepsilon
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \text{Max } h_0 \\ \text{subject to} \\ \varepsilon \frac{40u_1 + 55u_2 + 30u_3}{3v_1 + 5v_2} \\ \frac{45u_1 + 50u_2 + 40u_3}{2.5v_1 + 4.5v_2} \\ \dots\dots\dots \\ \text{and } u_1, u_2, v_1, v_2 \end{aligned}} \right\} \text{(M2)}$$

*The solution to the above model gives a value h_0 , the efficiency of depot 1, and the weights leading to that efficiency. **If $h_0 = 1$ then depot 1 is efficient relative to the others** but if h_0 turns out to be less than 1 then some other depot(s) is more efficient than depot 1, even when the weights are chosen to maximise depot 1's efficiency.*

تبدیل مدل کسری به مدل خطی

$$\left. \begin{array}{l} \text{Max } h_0 = \sum_r u_r y_{rj_0} \\ \text{subject to} \\ \sum_i v_i x_{ij_0} = 100 \text{ (say)} \\ \sum_r u_r y_{rj} - \sum_i v_i x_{ij} \leq 1 \quad j=1,2,\dots,n. \\ u_r, v_i \geq \varepsilon \end{array} \right\} \text{(M3)}$$

جواب مسئله

Depot	Efficiency
19	1.00
15	1.00
14	1.00
12	1.00
9	0.96
5	0.95
2	0.94
16	0.91
10	0.89
20	0.84
13	0.83
6	0.83
1	0.82
3	0.82
7	0.71
4	0.65
11	0.63
17	0.55
8	0.52
18	0.42

- At one extreme depot 18 has an efficiency of only 0.42. This can broadly be interpreted as saying that depot 18 should have been able to support its activity levels with only 42% of its resources.
- In fact 12 of the depots have an efficiency below 0.9 indicating a fair degree of discrimination

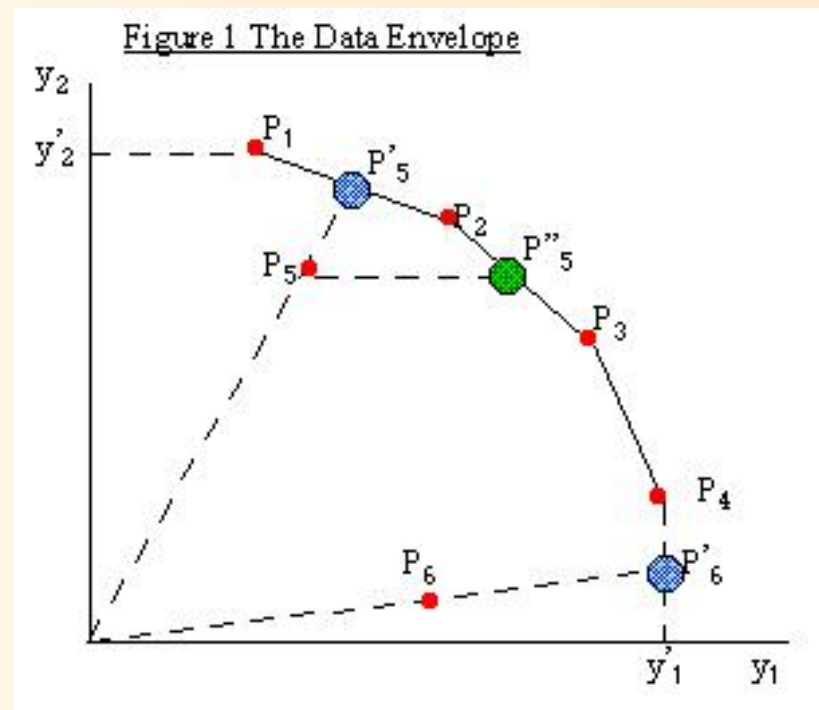
Targets for Depot 18 (efficiency 0.42)

ضرایب میزان بهینه منابع و خروجیها را
برای بهره وری محاسبه شده تعیین میکنند

Variable	Actual	Target
- stock	4.0	1.7
- wages	6.0	2.5
+ issues	25.0	25.3
+ receipts	38.0	38.0
+ reqs	20.0	20.0

مثال به روش گرافیکی

- a set of units P_1, P_2, \dots, P_6 with each unit consuming the same amount of a single resource and producing different amounts of outputs, y_1 and y_2
- For a given amount of resource input, units providing greater amounts of the outputs will be the efficient ones.
- Applying the DEA approach to this set of units will identify units P_1, P_2, P_3 and P_4 as efficient and they provide an envelope round the entire data set units P_5 and P_6 are within this envelope and are inefficient.
- The data envelope has been notionally extended to the axes by the lines P_1y_2' and P_4y_1' to enclose the data set.



For unit P_5 the peer group consists of the units P_1 and P_2 and a set of targets for P_5 is provided at P_5' .

تغییر مدل

Primal Model:

$$\text{Max } h_0 = \sum_r u_r y_{rj_0}$$

subject to

$$\sum_i v_i x_{ij_0} = 100 \text{ (say)}$$

$$\sum_r u_r y_{rj} - \sum_i v_i x_{ij} \leq 1 \quad j=1,2,\dots,n$$

$$-v_i \leq -\varepsilon \quad i=1,2,\dots,m$$

$$-u_r \leq -\varepsilon \quad r=1,2,\dots,t$$

dual variables

Z

λ_0

λ_j

s_i^+

s_r

(M4)

مدل دوگان

Dual Model:

$$\text{Min } 100 Z_0 - \varepsilon \sum_r s_r^+ - \varepsilon \sum_i s_i^-$$

subject to

$$x_{ij_0} Z_0 - s_i^- - \sum_j x_{ij} \lambda_j = 0 \quad i=1, \dots, m \quad (M5)$$

$$-s_r^+ + \sum_j y_{rj} \lambda_j = y_{rj_0} \quad r=1, \dots, t$$

$$\lambda_j, s_r^+, s_i^- \geq 0, Z_0 \text{ unconstrained.}$$

نکاتی در خصوص مسئله دوگان

- The first thing to note is that the primal model has $n + t + m + 1$ constraints whilst the dual model has $m + t$ constraints. As n , the number of units, is usually considerably larger than $t + m$, the number of inputs and outputs, it can be seen that the primal model will have many more constraints than the dual model. For linear programs in general the more constraints the more difficult a problem is to solve. Hence for this reason it is usual to solve the dual DEA model rather than the primal.
- The dual variables $\text{Lambda}(j)$ are thus also the shadow prices related to the constraints limiting the efficiency of each unit to be no greater than 1. It is also known that where a constraint is binding, a shadow price will be positive normally and where the constraint is non-binding the shadow price will be zero

مزایا و معایب

- it characterizes each DMU by a single efficiency score;
- by projecting inefficient units on the efficient envelop, it highlights areas of improvements for each single DMU;
- it facilitates making inferences on the DMUs' general profile
- the possibility of handling multiple inputs and outputs stated in different measurement units;
- the focus on a best-practice frontier, instead of on population central-tendencies. Every unit is compared to an efficient unit or a combination of efficient units
- no restrictions are imposed on the functional form relating inputs to outputs
- limitations in aggregating different aspects of efficiency, especially in the case where DMUs perform multiple activities.
- insensitivity to intangible and categorical components (for instance, the service quality in a bank branch setting).
- Another problem relates to the difficulty of mixing different dimensions of the analysis
- the impossibility of ranking efficient units; indeed all efficient units have an efficiency score of 100%;
- from a managerial point of view, it may be more useful to compare branches to a frontier of absolute best performance

برنامه ریزی استراتژیک و DEA

Profit	High	Under-performing potential stars	Benchmark group
	Low	Problem Branches	Candidates for divestiture
		Low	High

Efficiency

مسئله

SUMMARY OF OUTPUTS AND INPUTS FOR BURGER king

Service unit	Meals sold	Labor-hours	Material dollars
1	100	2	200
2	100	4	150
3	100	4	100
4	100	6	100
5	100	8	80
6	100	10	50

Burger KING Productivity Frontier

